The Addition Formulas in Trigonometry

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Why not the usual?

- In Mathematics, we know that the distributive property allows 7(x + 5) = 7x + 35
- With derivatives, $(f+g)' = f' + g'$
- With integrals, $\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx$

So it seems that we can start to generalize a concept like this in some areas.

Linear Transformations

- A **Linear Transformation** (or linear map) is a special type of function where:
  - $F(x + y) = F(x) + F(y)$ and
  - $F(kx) = kF(x)$ for a constant/scalar k.

These functions are extensively studied in Linear Algebra (Math 270) and get their name by always mapping a line into a line.

Not all functions are linear transformations, which means we need to do a bit more examination.

What else could it be?

- With other areas of mathematics, we saw that
  - $\log(x) + \log(y) \neq \log(x + y)$
  - $(x+y)^2 \neq x^2 + y^2$
  - $(x+y)^3 \neq x^3 + y^3$
  - $\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$

  So what should we do with $\cos(x + y)$ and $\sin(x + y)$?

First attempt at Linear Transformation

Does $\cos(x + y) = \cos(x) + \cos(y)$?

Try with some known values:

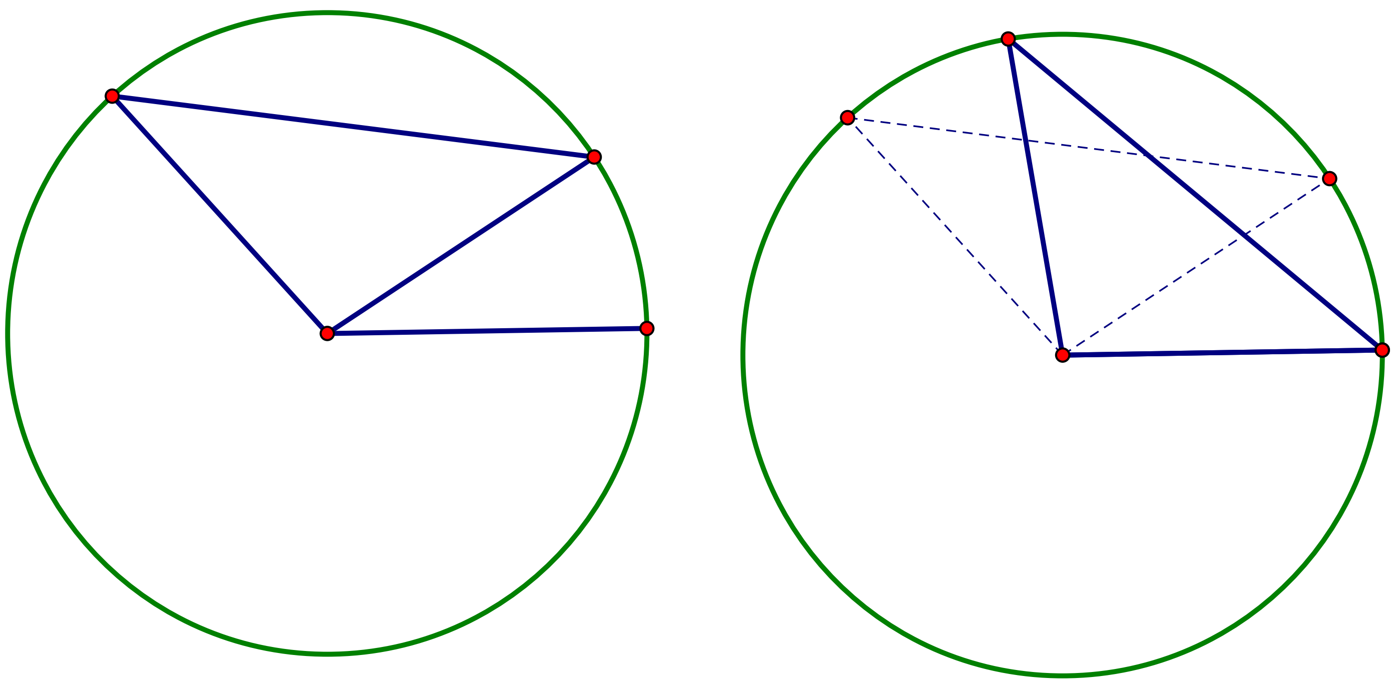
$$\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$$

$$\cos\left(\frac{3\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$$

Finish this result to see if it checks out.

So that failed… let's try distance!

Find $\cos(x - y)$ based on the unit circle.



So that failed… let's try distance! (2)

Find $\cos(x - y)$ based on the unit circle. Label the coordinates of each point.

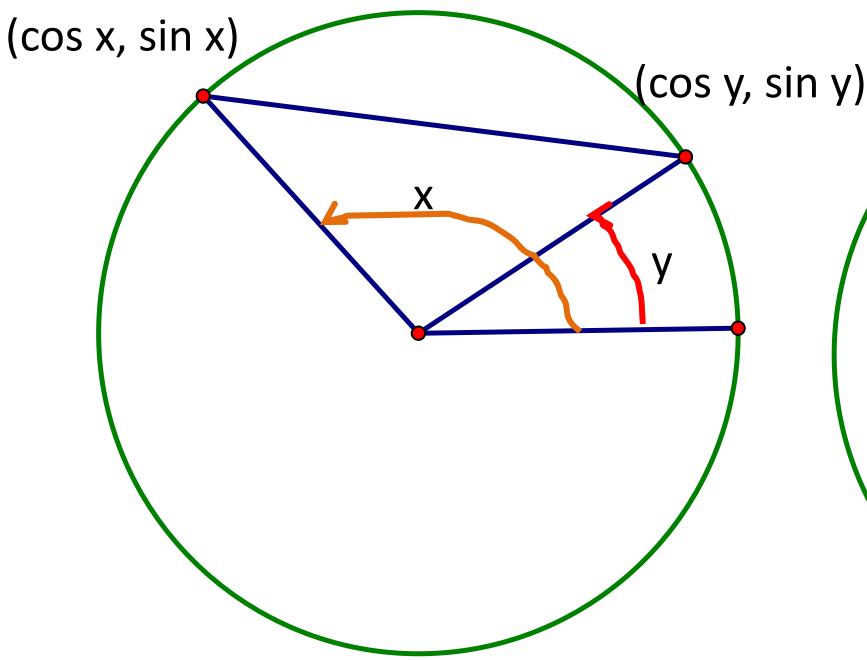


Figure 1

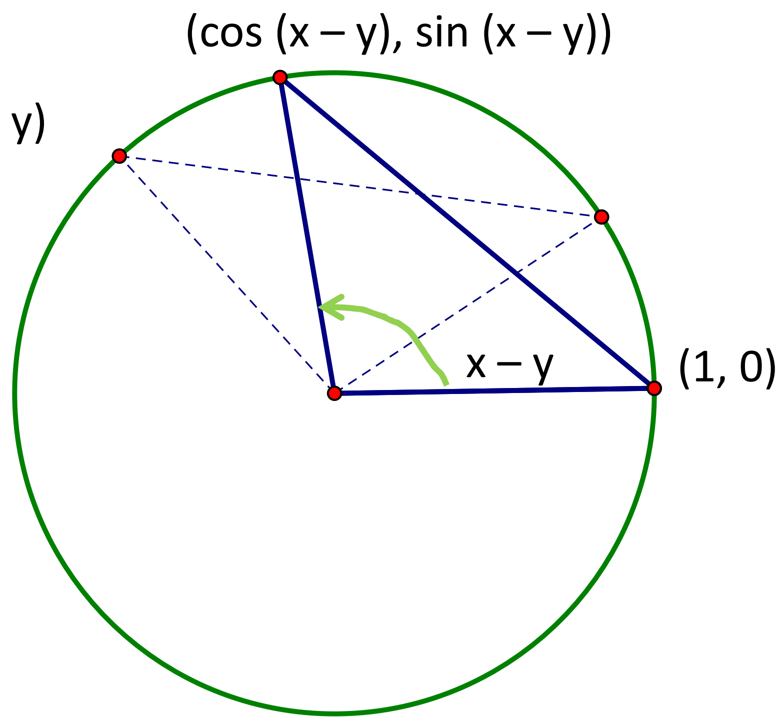


Figure 2

Find $\cos(x - y)$ based on the unit circle.

Distance between the two labeled points in Figure 1.

$$d = \sqrt{(cosx - cosy)^2 + (sinx - siny)^2}$$
$$d = \sqrt{cos^2x - 2cosxcosy + cos^2y + sin^2x - 2sinxsiny + sin^2y}$$
$$d = \sqrt{(cos^2x + sin^2x) + (cos^2y + sin^2y) - 2cosxcosy - 2sinxsiny}$$
$$d = \sqrt{1 + 1 - 2cosxcosy - 2sinxsiny}$$
$$d = \sqrt{2 - 2cosxcosy - 2sinxsiny}$$
$$d = \sqrt{2 - 2(cosxcosy + sinxsiny)}$$

Keep this in mind as we move to the next part.

Find $\cos(x - y)$ based on the unit circle.

Distance between the two labeled points in Figure 2.

$$d = \sqrt{(cos(x-y)-1)^2 + (sin(x-y)-0)^2}$$
$$d = \sqrt{cos^2(x-y) - 2cos(x-y) + 1 + sin^2(x-y)}$$
$$d = \sqrt{\left(cos^2(x-y) + sin^2(x-y)\right) + 1 - 2cos(x-y)}$$
$$d = \sqrt{1 + 1 - 2cos(x-y)} = \sqrt{2 - 2cos(x-y)}$$

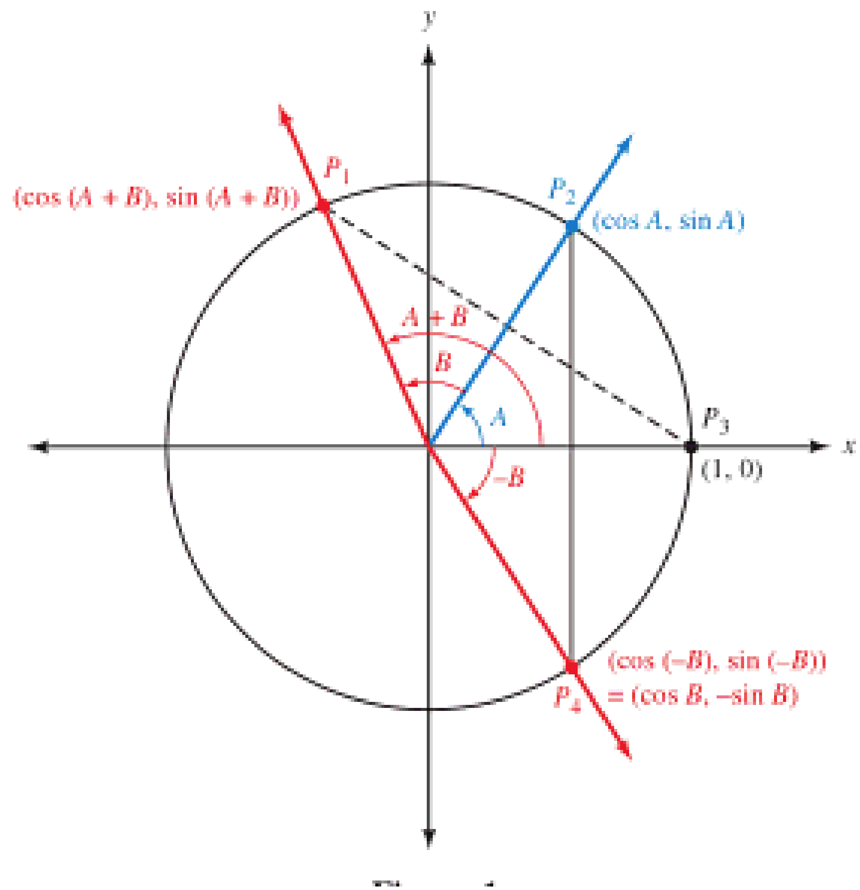
Now compare this to the previous distance: $d = \sqrt{2 - 2(cosxcosy + sinxsiny)}$

Since the distances must be the same, we conclude that:
$cos(x-y) = cosxcosy + sinxsiny$

Want $sin(x-y)$, then just check the complement:
$sin(x-y) = cos\left(\frac{\pi}{2} - (x-y)\right) = cos\left(\left(\frac{\pi}{2} - x\right) - (-y)\right).$

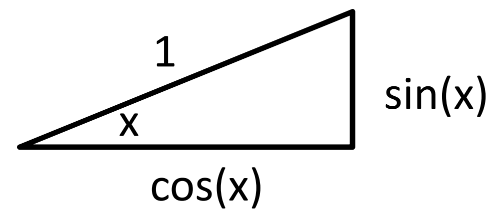
What about the other ones?

You can use a similar picture to graph addition, but in this case, you'll need to think of a clockwise rotation of an angle, so one angle will be negative.

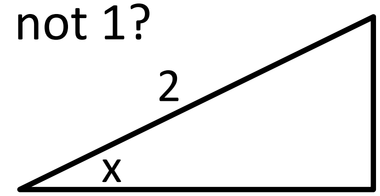


Trigonometry review

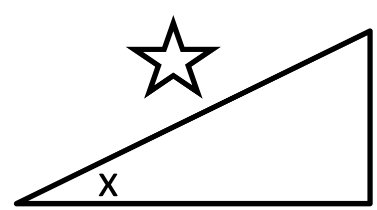
We’ll take a look at standard (unit-circle) triangles and expand it.



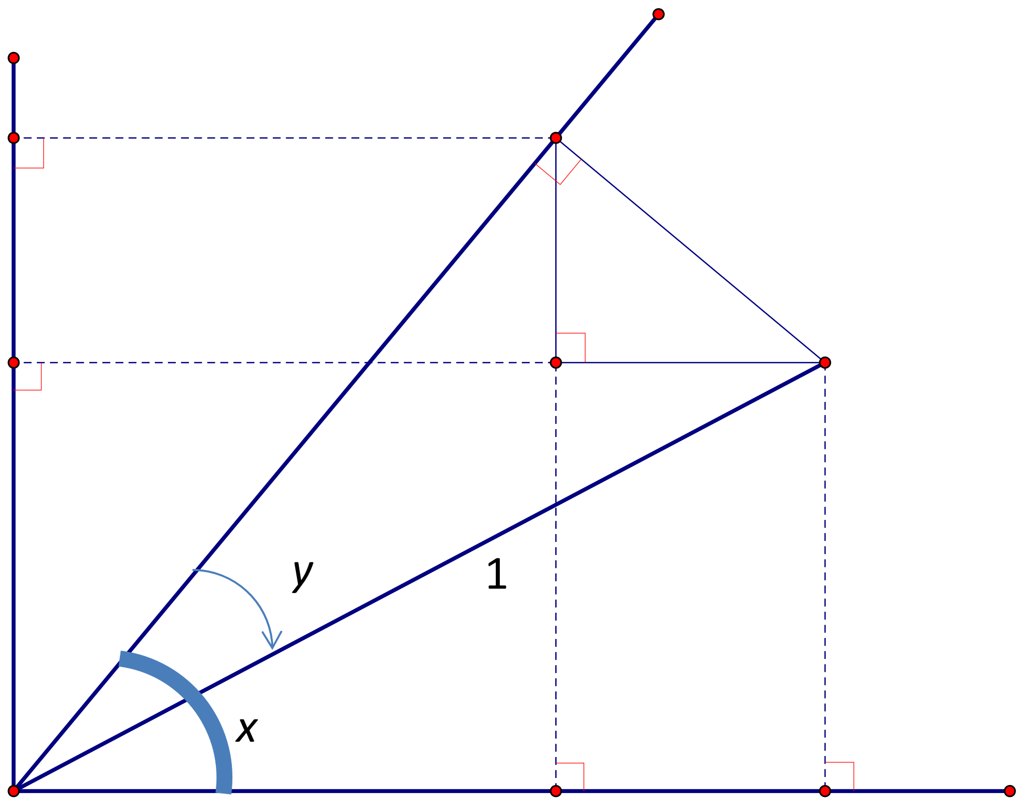
What happens when the hypotenuse (radius) is not 1?



Let’s try one more just to see if we’ve got it down. What would be the lengths of the sides in this picture?



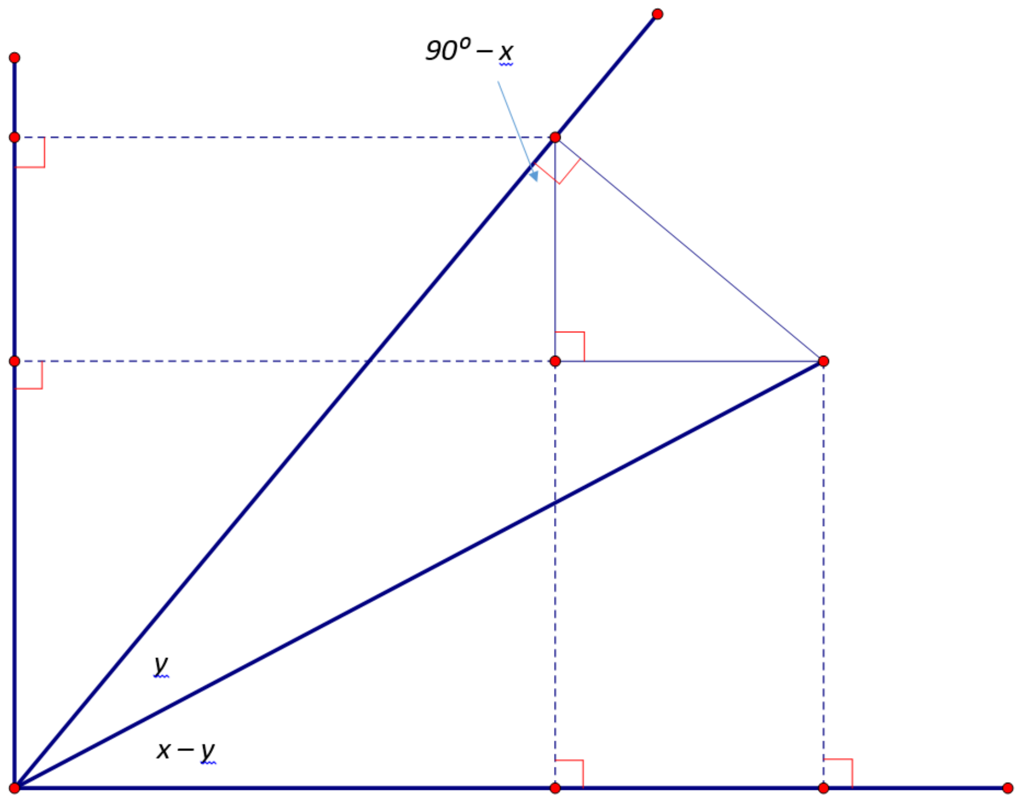
Picture version – with a twist! (1)



Draw an angle, x, and extend it for a while. Then from the same vertex, draw angle y down from x (clockwise). Extend the length to be 1. From the ray that created angle x, construct a right angle that goes through the endpoint creating angle y. Finish by drawing parallel and perpendicular lines as needed. Then label the sides using your trigonometric knowledge from the previous slide.

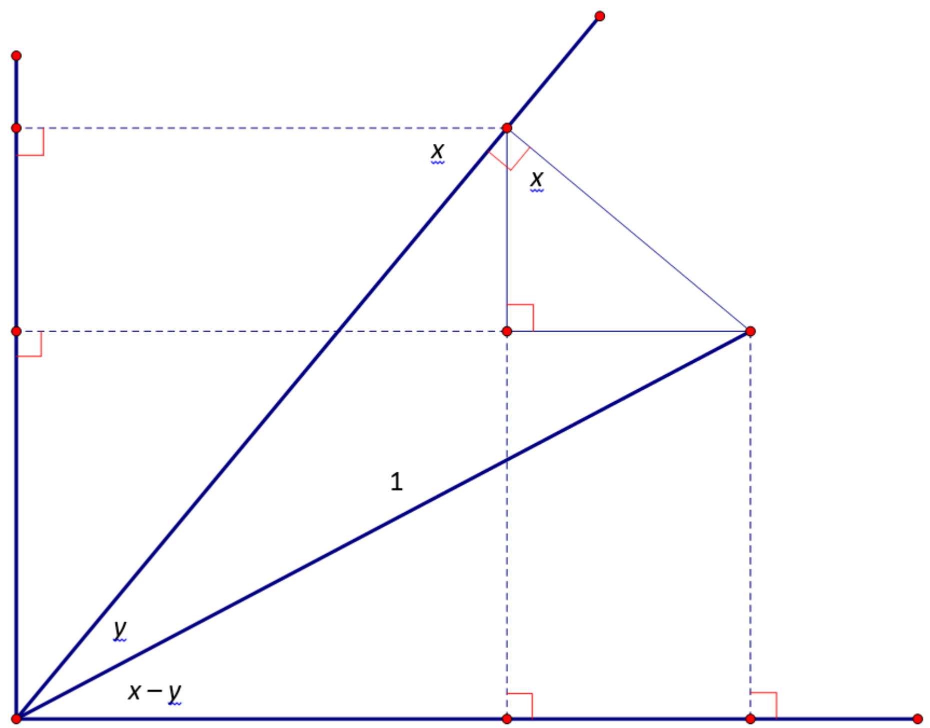
It's kind of fun, too! ☺

Picture version – with a twist! (2)



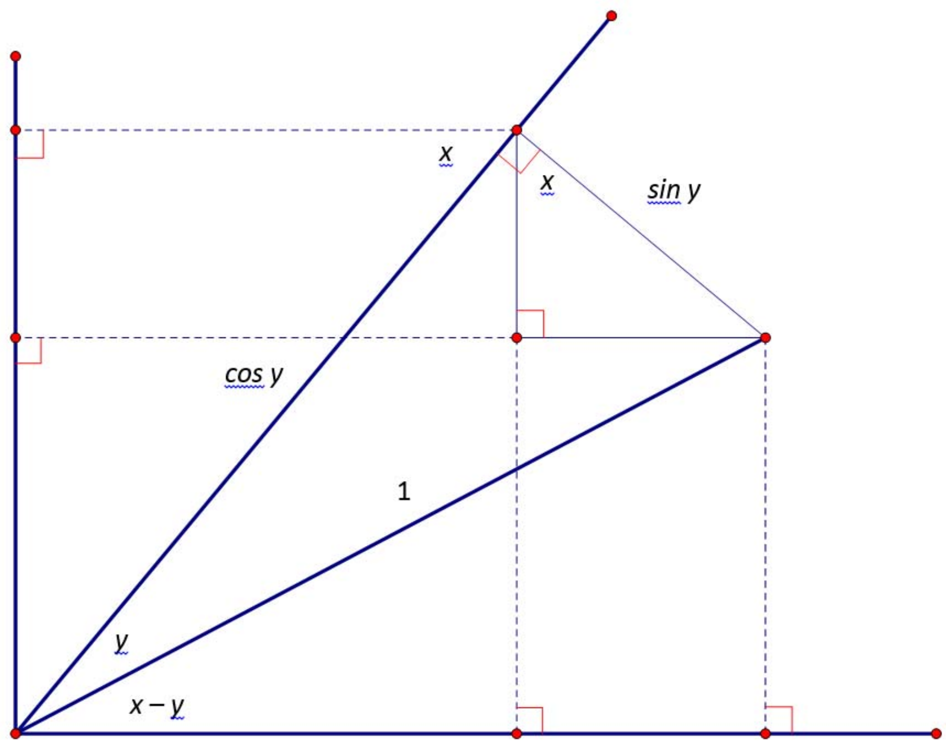
The right triangle formed with the bottom and angle x creates another angle, 90° – x. Label this angle.

Picture version – with a twist! (3)



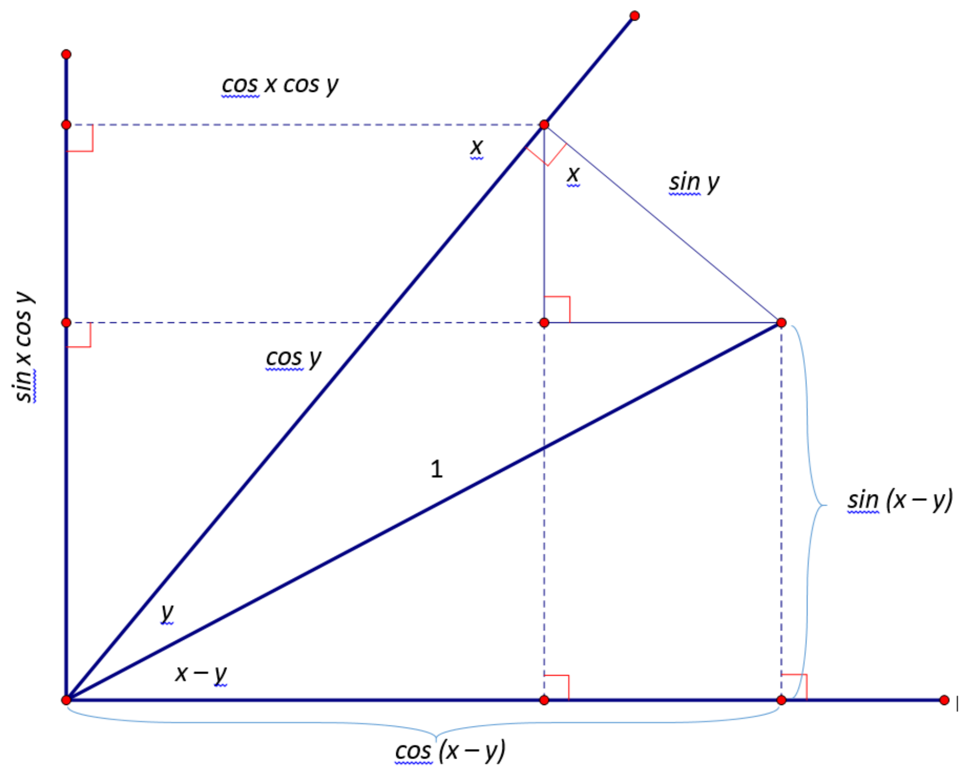
Using complementary angles and alternate interior angles from parallel lines, label the other x angles in the picture. These will be extremely helpful in the next few steps. Notice that the only length labeled is "1" currently.

Picture version – with a twist! (4)



Now, using the right triangle with y as one angle and 1 as the hypotenuse, label the other two sides. This is effectively what you would do with a unit circle triangle, but in this case, there is no circle.

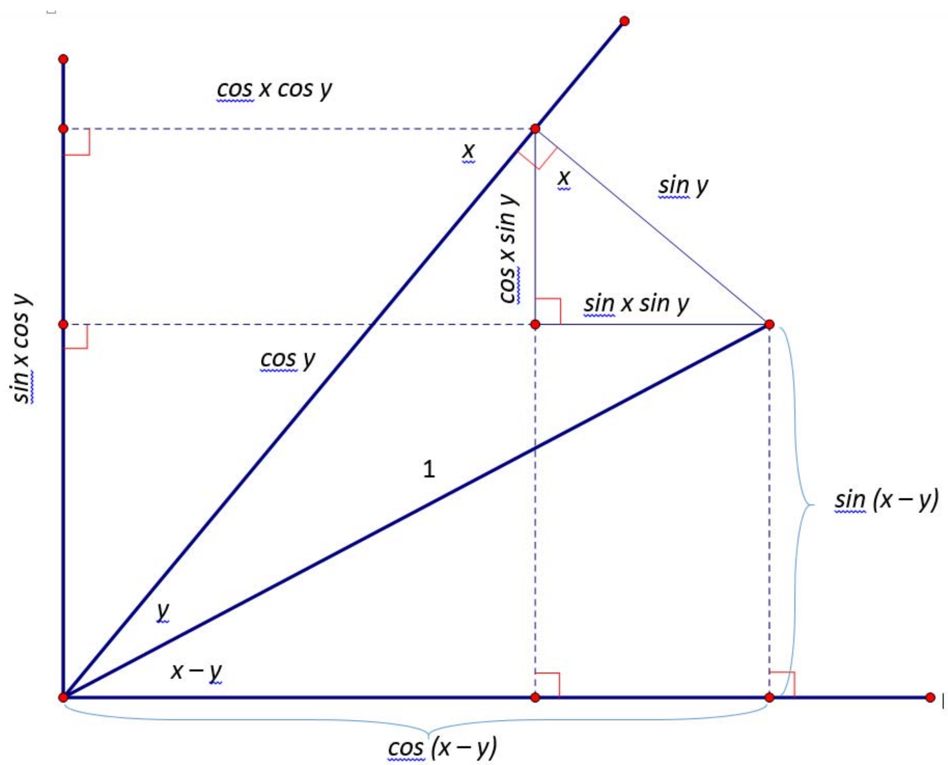
Picture version – with a twist! (5)



Now, using the right triangle with x as one angle (top left), the one with a hypotenuse of cos y, label the other two sides. And while we're at it, let's label the bottom right triangle with an angle of x – y that has hypotenuse of 1 as well. You'll notice that the picture is getting rather busy… but we need all these pieces.

Almost done!

Picture version – with a twist! (6)



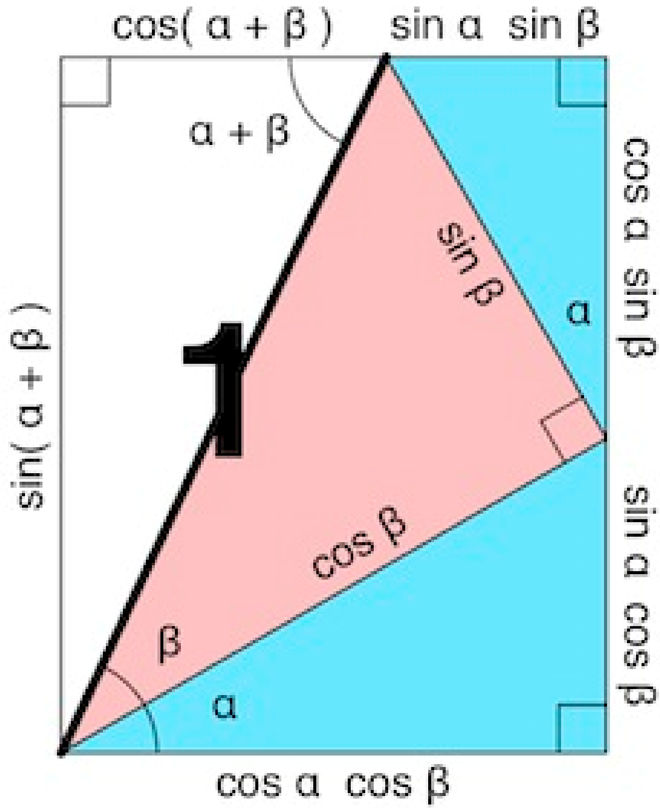
Lastly, label the top right triangle with an angle of x and hypotenuse of sin y.

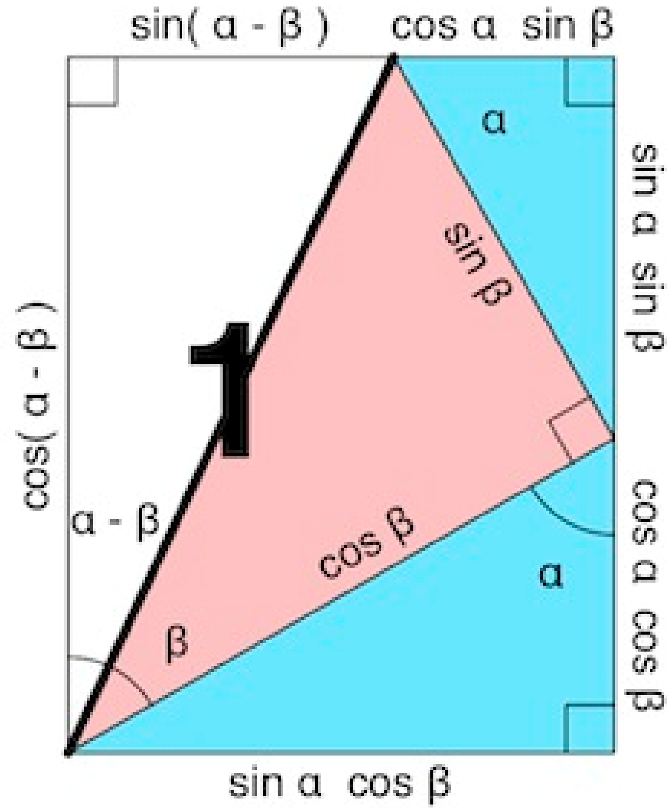
Phew. All done.

Do you see the identities for cos(x – y) and sin(x – y) in the picture?

Look at the lengths of horizontal and perpendicular sides. ☺

Other pictures





$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

There are many other techniques

Many of the other methods can involve distance or pictures, but rely on the law of cosines.

But is there another way… a way that doesn't require long arithmetic, memorizing or remembering some picture, or remembering some method to create a picture that would work?

Euler to the Rescue

Euler's (trigonometric) formula:
$e^{ix} = \cos(x) + i\sin(x)$

This equation comes from rotating a line in the complex plane. We can't do any manipulation on the complex portion to turn it into the real portion. If we have two equivalent complex numbers, their real parts must be the same and the complex parts must be the same.

Now that we have the basis, let's check how it works (calculators out if possible):

- $e^{i\left(\frac{\pi}{3}\right)} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. This corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.
- $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1.$

Rewriting this and you'll get: $e^{i\pi} = -1 \rightarrow e^{i\pi} + 1 = 0.$ This is one incredible formula as it includes the five most important constants in all of mathematics together with no additional coefficients.

- e, the natural number
- i, the imaginary number
- π, the constant pi related to a circle
- 1, the additive identity
- 0, the multiplicative identity

Richard Feynman, an incredibly famous physicist, claimed this was the jewel of mathematics. Some have written that because of the innate beauty and simplicity of this equation, that Euler used it as a proof that god must exist.

Why is this formula so powerful?

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y) \qquad (1)$$

$$e^{i(x+y)} = e^{ix}e^{iy}$$

$$e^{i(x+y)} = \big(\cos(x) + i\sin(x)\big)\big(\cos(y) + i\sin(y)\big)$$

$$e^{i(x+y)} = \cos(x)\cos(y) + i\sin(x)\cos(y) + i\sin(y)\cos(x) + i^2\sin(x)\sin(y)$$

$$e^{i(x+y)} = \cos(x)\cos(y) + i\sin(x)\cos(y) + i\sin(y)\cos(x) - \sin(x)\sin(y)$$

$$e^{i(x+y)} = \cos(x)\cos(y) - \sin(x)\sin(y) + i\big(\sin(x)\cos(y) + \sin(y)\cos(x)\big) \qquad (2)$$

At this point, we will need to equate the real parts and the imaginary parts of the two equations.

Why is this formula so powerful? (2)

$$e^{i(x+y)} = \boxed{\cos(x+y)} + i\ \boxed{\sin(x+y)} \quad (1)$$

$$e^{i(x+y)} = \boxed{\cos(x)\cos(y) - \sin(x)\sin(y)} +$$

$$i\boxed{\left(\sin(x)\cos(y) + \sin(y)\cos(x)\right)} \qquad (2)$$

Once you do this, you can see two identities with just one quick expansion of the distributive property:

- Red (Real): $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- Green (Imaginary): $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

You could continue doing this with $e^{i(x-y)}$ as well; no pictures needed, and very little algebra. As an added bonus, the derivation is extremely quick and you get two formulas each time!

Why is this formula so powerful?

What about other identities? Maybe Even/Odd?

$e^{i(-x)} = \cos(-x) + i\sin(-x)$

$e^{i(-x)} = e^{-ix} = \dfrac{1}{e^{ix}} = \dfrac{1}{\cos(x) + i\sin(x)}$

Multiply by the complex conjugate.

$$\frac{1}{\cos(x) + i\sin(x)}\left(\frac{\cos(x) - i\sin(x)}{\cos(x) - i\sin(x)}\right)$$

$= \dfrac{\cos(x) - i\sin(x)}{\cos^2(x) + \sin^2(x)} = \cos(x) + i\big(-\sin(x)\big)$

So this means that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$

No pictures… just a little algebra!

Could we do more?

Heck yes! We could use this to find the sum to product formulas, but they require the ability to remember a substitution in the middle of the problem. These types of substitutions do come up in calculus, but we'll leave them off for now.

What kind of substitutions?

When you look at x, you probably don't immediately think $x = x + 0$.

Further, you probably don't think $x + 0 = x + \frac{y}{2} - \frac{y}{2}$, right?

Oh, and once you do think about these, do you immediately think:
$x + \frac{y}{2} - \frac{y}{2} = \frac{x}{2} + \frac{x}{2} + \frac{y}{2} - \frac{y}{2} = \frac{x}{2} + \frac{y}{2} + \frac{x}{2} - \frac{y}{2} = \frac{x+y}{2} + \frac{x-y}{2}$. ☹

Well, those are the types of substitutions needed for the sum-to-product formulas.

Why use it?

Euler's formula is a tool you can use, but don't have to. Like many things in math, it is extremely useful at times and not as useful in other situations.

Try it out. As you get comfortable with it, you may find some very interesting results!

Added bonus, this opens up a whole new world. Indeed, if $e^{i\pi} = -1$, which we confirmed that it did on an earlier slide, then we could rewrite this using logarithms: $e^{i\pi} = -1 \leftrightarrow ln(-1) = i\pi$

So logarithms could actually be defined over negative numbers if we allowed complex number outputs. Test this on your calculator in complex mode to see the result. Back when covering logarithms, the domain was restricted to all non-negative real numbers… but that was needed to get the result to be a real number. Expanding our definition/domain will allow us to MORE, not less. Now, go explore!

How Important are the Addition Formulas?

Back in the 1950s, Professor Hans Rademacher* showed that all of trigonometry could be developed with just two functions that he called "C" and "S" as long as:

1. $C(x-y) = C(x)C(y) + S(x)S(y)$
2. $S(x-y) = S(x)C(y) - S(x)C(y)$
3. $\lim\limits_{x\to 0^+} \frac{S(x)}{x} = 1$

Those of you in calculus may recognize (3) as one of the most important limits in calculus, necessary for the formulation of trigonometric derivatives.

*Mathematics Teacher Vol L (January 1957) pp. 45-48.

Calc link (power reduction) - 1

$$\int cos^2(x)dx = \int \frac{cos(2x)+1}{2}dx$$

$e^{ix} = \cos(x) + i\sin(x)$
$e^{-ix} = \cos(x) - i\sin(x)$

So $e^{ix} + e^{-ix} = 2\cos(x)$ which means $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$

This means $cos^2(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 = \frac{e^{i2x}+2+e^{-i2x}}{4} = \frac{2cos(2x)+2}{4}$

Calc link (power reduction) - 2

Expanding the technique from the previous page, we can even do something like this:

$$\int cos^6(x)dx = \int \frac{2cos(6x) + 12cos(4x) + 30cos(2x) + 20}{64} dx$$

It helps to remember Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Calc link (power reduction) - 3

$\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, which means $e^{ix} + e^{-ix} = 2\cos(x)$.

$$\cos^6(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^6 = \frac{\left(e^{ix} + e^{-ix}\right)^6}{2^6}$$

Using Pascal's Triangle for coefficients:
$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2\,b^4 + 6ab^5 + b^6$

Expanding the binomial:

$$\left(e^{ix} + e^{-ix}\right)^6 = \left(e^{ix}\right)^6 + 6\left(e^{ix}\right)^5\left(e^{-ix}\right) + 15\left(e^{ix}\right)^4\left(e^{-ix}\right)^2 + \\ 20\left(e^{ix}\right)^3\left(e^{-ix}\right)^3 + 15\left(e^{ix}\right)^2\left(e^{-ix}\right)^4 + 6\left(e^{ix}\right)\left(e^{-ix}\right)^5 + \left(e^{-ix}\right)^6$$

Simplifying exponents:

$$\left(e^{ix} + e^{-ix}\right)^6 = e^{i6x} + 6e^{i5x}\left(e^{-ix}\right) + 15e^{i4x}\left(e^{-i2x}\right) + \\ 20e^{i3x}\left(e^{-i3x}\right) + 15e^{i2x}\left(e^{-i4x}\right) + 6e^{ix}\left(e^{-i5x}\right) + e^{-i6x}$$

Calc link (power reduction) - 4

Simplifying exponents:

$$\left(e^{ix}+e^{-ix}\right)^6 = e^{i6x}+e^{-i6x}+6e^{i4x}+6e^{-i4x}+15e^{i2x}+15e^{-i2x}+20e^{i0x}$$

Grouping like objects:

$$\left(e^{ix}+e^{-ix}\right)^6 = \left(e^{i6x}+e^{-i6x}\right)+6\left(e^{i4x}+e^{-i4x}\right)+15\left(e^{i2x}+15e^{-i2x}\right)+20$$

Euler's formula then substitutes since $e^{ix}+e^{-ix}=2\cos(x)$:

$$\left(e^{ix}+e^{-ix}\right)^6 = 2cos6x+6(2cos4x)+15(2cos2x)+20$$

This shows why $cos^6(x)=\frac{2cos(6x)+12cos(4x)+30cos(2x)+20}{64}$.

And now you can do the integration in calculus II much quicker:

$$\int cos^6(x)dx = \int \frac{2cos(6x)+12cos(4x)+30cos(2x)+20}{64}dx$$

Further links

Those in Calculus 2 will see series. You could confirm Euler's formula using using Maclaurin series for each function $e^{ix} = \cos(x) + i\sin(x)$.

In Pre-Calculus 2 (Math 131), a useful result is that every complex number can be expressed in polar coordinates.

Another Pre-Calculus 2 result is DeMoivre's Formula that can be extended and used to find complex roots of numbers: $\left(e^{ix}\right)^n = \cos(nx) + i\sin(nx)$

Those in Discrete Math (Math 226) could prove that the previous formula is true for all values of *n*.

Calc I and II (Math 150 and 155) often have to substitute in power reduction formulas for powers of trigonometric functions.