IMPROBABLE PROBABILITIES

WHAT TO DO WHEN WHAT YOU THINK SHOULD HAPPEN DOESN'T

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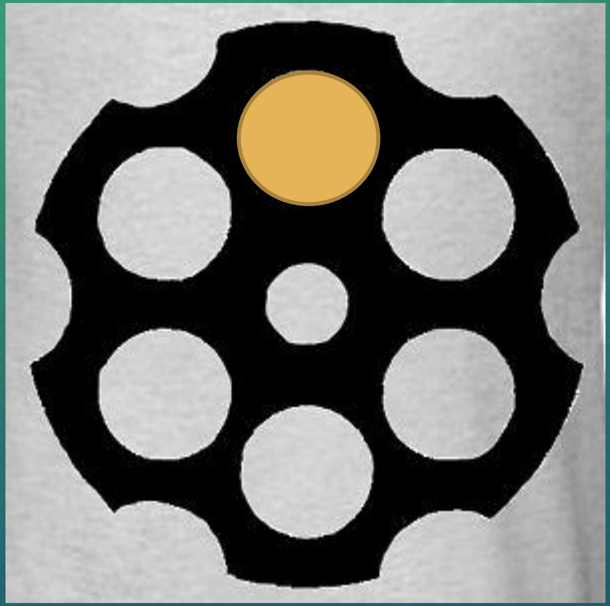
MATHEMATICAL DUELING

Duels have existed for a long time and to avoid any actual violence, let's settle on Nerf gun duels. For those opposed to even Nerf gun duels, imagine two people shooting at a target – the first person to hit the target requires the other person to do the chore. My kids would enjoy this approach as there is one chore that they despise… cleaning the toilet.

We'll start with a Nerf revolver gun that has 1 dart in it. The cylinder is spun and handed to the first child. They aim at the target and pull the trigger. If they hit, the other person has to clean the toilet.

SIDE NOTE: We started this tradition with the more pure idea that if you hit the target, you had to do the chore. As you may imagine, this ended up with an event that lasted a very long time. Time to "switch horses." In retrospect, both of these involve hitting the target which is a bit of a problem since one daughter has practiced a lot with Nerf guns to avoid the toilet. Perhaps the best way to approach this now is that the person who has the gun fire the dart has to do the toilet. ☺ Now it's fair again!

ONE DART - PROBABILITIES

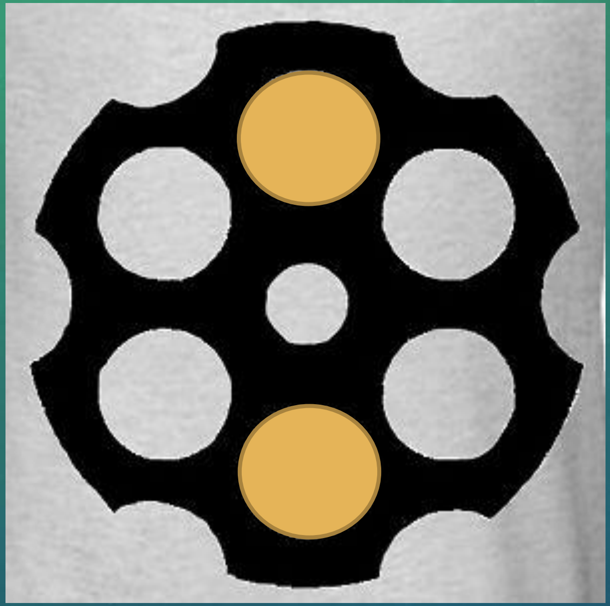


We will start with one Nerf dart in the gun. Imagine that this cylinder is then spun around and ends up randomly on one of the six spaces.

What is the probability that the dart will fire for the first child? Remember that for this situation, if the dart fires, you have to clean my bathroom.

Ok. Now assume that it didn't fire, so the gun is passed to the next child. If you were this child, would you rather spin the cylinder again, or would you rather just pull the trigger?

TWO DART – PROBABILITIES – PART 1: SPLIT DARTS

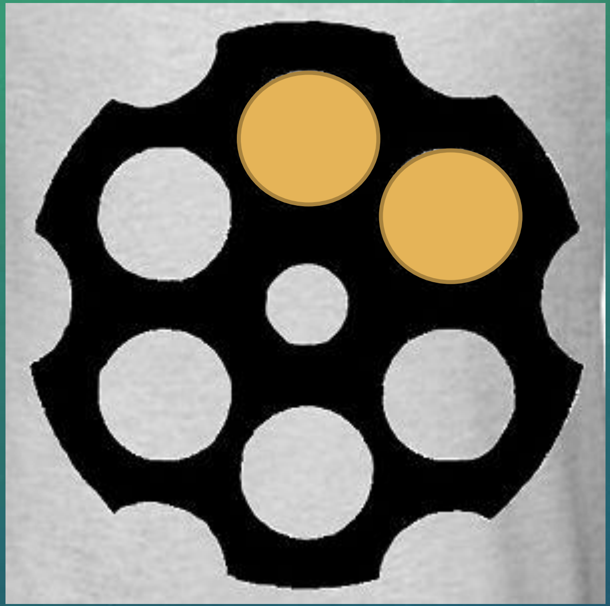


We will start with two Nerf darts in the gun and they are not next to each other. Imagine that this cylinder is then spun around and ends up randomly on one of the six spaces.

What is the probability that the dart will fire for the first child? Remember that for this situation, if the dart fires, you have to clean my bathroom.

Ok. Now assume that it didn't fire, so the gun is passed to the next child. If you were this child, would you rather spin the cylinder again, or would you rather just pull the trigger?

TWO DART – PROBABILITIES PART 2: KEEP IT TOGETHER

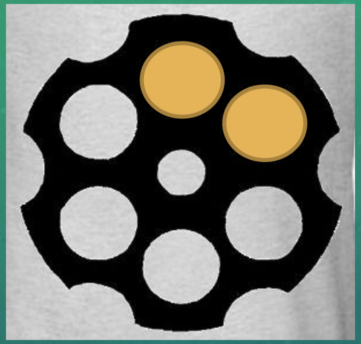


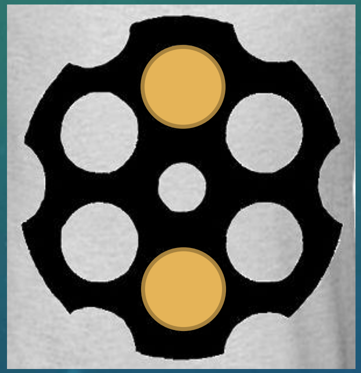
We will start with two Nerf darts in the gun and they ARE next to each other. Imagine that this cylinder is then spun around and ends up randomly on one of the six spaces.

What is the probability that the dart will fire for the first child? Remember that for this situation, if the dart fires, you have to clean my bathroom.

Ok. Now assume that it didn't fire, so the gun is passed to the next child. If you were this child, would you rather spin the cylinder again, or would you rather just pull the trigger?

WHAT JUST HAPPENED?





Both situations have 2 darts and 4 open cylinders. Yet the probabilities are not the same. Why?

Would this extend if we went further. What if we put in…

- 3 darts. Better to spin or just pull?
- 4 darts. Better to spin or just pull?
- 5 darts. Better to spin or just pull?

For these, would it matter if the darts were next to each other or not?

OKAY – NERF DUELS.

2 people get a Nerf gun. Each has a specific chance of hitting, but not great. First person to hit the other person with a Nerf dart wins.

Person A has a 30% chance of hitting Person B, while Person B has a 10% chance of hitting Person A.

If Person A goes first…

- what is the probability that Person A will win?
- what is the probability that Person B will win?

If Person B goes first…

- what is the probability that Person A will win?
- what is the probability that Person B will win?

With A having 3 times the chance of hitting B, will A have 3 times the probability of winning?

NERF DUEL SOLUTION – PART 1

$30\% = 0.3$, so here's how it would look math-wise:

Probability that Person A wins if Person A goes first is:
$P(A) = (0.3) + (0.7)(0.9)(0.3) + (0.7)^2(0.9)^2(0.3) + (0.7)^3(0.9)^3(0.3) + \cdots$

Probability that Person B wins if Person A goes first is:
$P(B) = (0.7)(0.1) + (0.7)^2(0.9)(0.1) + (0.7)^3(0.9)^2(0.1) + (0.7)^4(0.9)^3(0.1) + \cdots$

Here comes the math fun. These are both sums of geometric sequences, so the sum could be found using the standard formula from Pre-Calculus or Calculus 2: $S = \frac{a}{1-r}$. But for those of you not in those classes, it's hard to see what to do without more information. The tip is that $P(A) + P(B) = 1$.

$P(B)$ can be multiplied by 0.9 and then multiplied by 3 and we'll be so very close to $P(A)$… just need to add 0.3. So $P(A) = 3(0.9)P(B) + 0.3$. Just substitute that back into $P(A) + P(B) = 1$ and we're good:

$P(A) + P(B) = 1 \rightarrow 3(0.9)P(B) + 0.3 + P(B) = 1 \rightarrow 3.7P(B) = 0.7 \rightarrow P(B) = \frac{0.7}{3.7} = \frac{7}{37} = 0.\overline{189} \approx 19\%.$

NERF DUEL SOLUTION – PART 2

$P(A) + P(B) = 1 \rightarrow 3(0.9)P(B) + 0.3 + P(B) = 1 \rightarrow 3.7P(B) = 0.7 \rightarrow P(B) = \frac{0.7}{3.7} = \frac{7}{37} = 0.\overline{189} \approx 19\%.$

If $P(B) = \frac{7}{37}$, then since the two sum to 1, $P(A) = \frac{30}{37} = 0.\overline{810} \approx 81\%.$

Both of these were found assuming that player A went first. Since they hit on their first shot 30% of the time, it makes sense that they would win… but A wins more than 4 times as often, not 3 times like it may have seemed.

Of course, this was when A went first. Perhaps it would be fair to have B go first because they are not as good of a shot.

What do you think the chances are that B will win now? Better or worse than 19%?

NERF DUEL SOLUTION – PART 3

Probability that Person B wins if Person B goes first is:
$P(B) = (0.1) + (0.9)(0.7)(0.1) + (0.9)^2(0.7)^2(0.1) + (0.9)^3(0.7)^3(0.1) + \cdots$

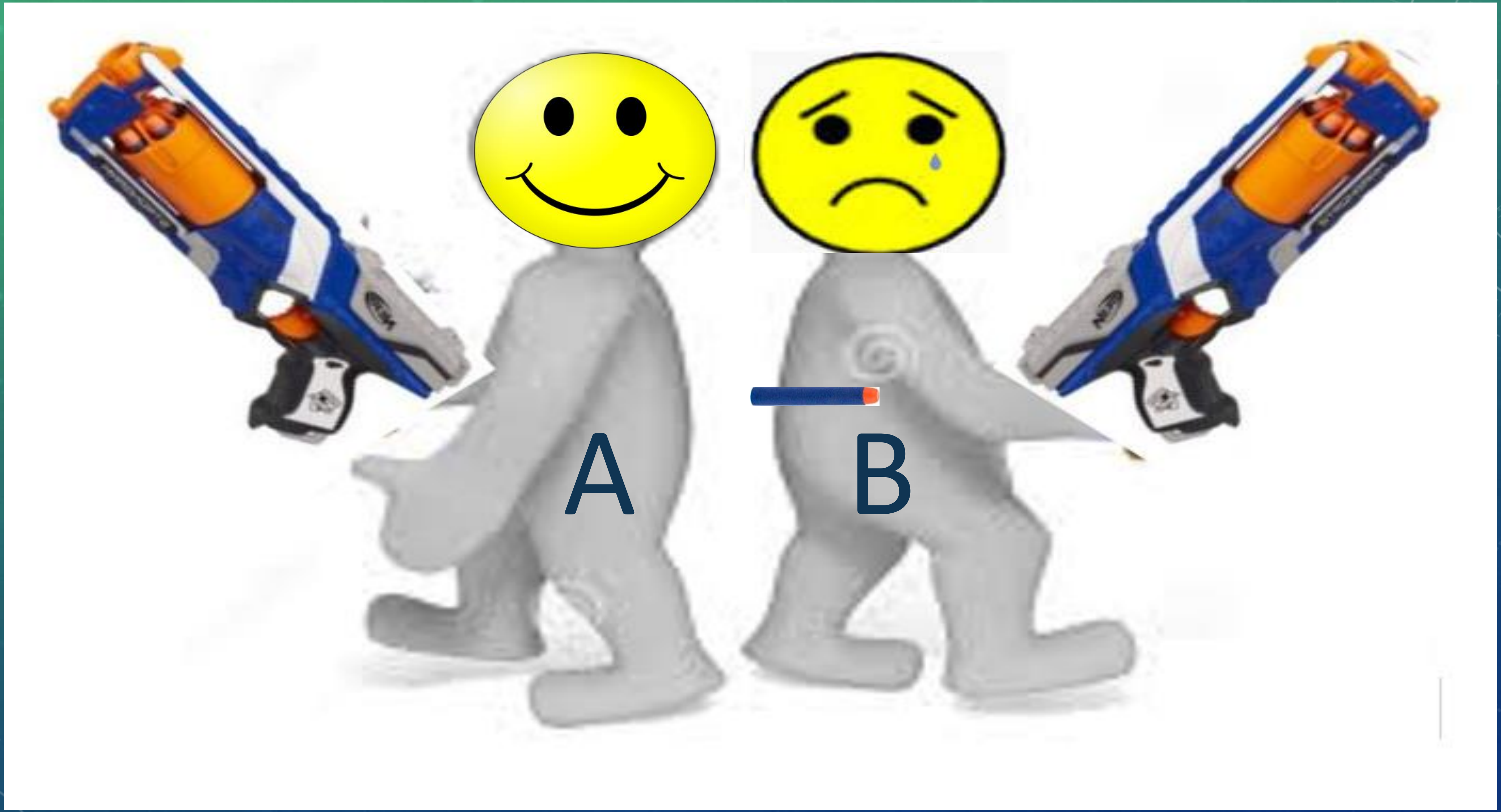
Probability that Person A wins if Person B goes first is:
$P(A) = (0.9)(0.3) + (0.9)^2(0.7)(0.3) + (0.9)^3(0.7)^2(0.3) + (0.9)^4(0.7)^3(0.3) + \cdots$

You could still use the formula: $S = \frac{a}{1-r}$. I like the other method here using: $P(A) + P(B) = 1$.

$P(B)$ can be multiplied by 0.9 and then multiplied by 3 and we'll be exactly to $P(A)$.
So $P(A) = 3(0.9)P(B)$. Just substitute that back into $P(A) + P(B) = 1$ and we're good:

$P(A) + P(B) = 1 \rightarrow 3(0.9)P(B) + P(B) = 1 \rightarrow 3.7P(B) = 1 \rightarrow P(B) = \frac{1}{3.7} = \frac{10}{37} = 0.\overline{270} \approx 27\%$.

$P(A) = \frac{27}{37} = 0.\overline{729} \approx 73\%$. This is just a little bit less than 3 times.. And no matter what, Person B is in trouble!


A
B

MENAGE-A-NERF? WHAT ABOUT 3 PEOPLE IN A DUEL?

So picture 3 people now, A, B, and C. We'll just use the numbers from before for A and B, but we'll say that Person C is a deadshot with Nerf… hitting 100% of the time. Truel rules:

- Person who goes first picks the direction.
- If A goes first, it could be A → B → C → A… or A → C → B → A…
- If needed, the order must stay the same as the first round.

If you were person C, it's pretty easy to determine the probabilities because you hit 100% of the time.

- If you shoot at A, then B has 1 shot at you (10% hit rate). So you will win 90% of the time.
- If you shoot at B, then A has 1 shot at you (30% hit rate). So you will win 70% of the time.
- Best choice is to shoot at A.

Person A is similar:

- If A shoots at C and hit (30% of the time), then we can use the previous work to determine who wins between A and B. A wins 73% of the time and B wins 27% of the time.
- If A shoots at C and misses, then C will shoot B and A has one more shot.

Why is A shooting at B with the first shot a bad idea?

HELP THE LITTLE GUY!

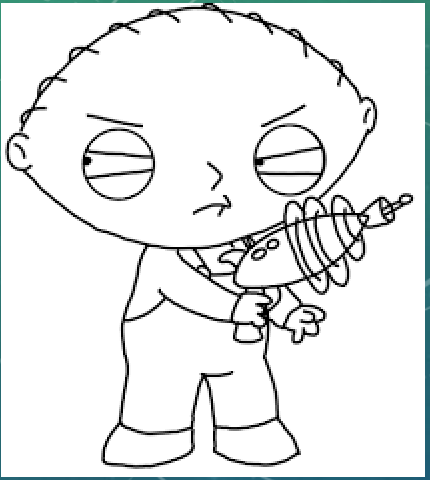
Now imagine that you are person B… not so good at this game, but you still want the best chance of winning.

You hit only 10% of the time. If you shoot at B, you'll pretty much miss. Your hope is that B shoots C and you have a real shot at B again.

But if B misses, then you're toast.

B needs your help. What is the best strategy for B?

LITTLE GUY SOLUTION.



Well, things aren't looking good. But here we go.

Person B shoots at A first.

- If B hits A, then C will shoot B and it's all over. That's 10% of the time… B loses quickly.
- If B misses A (90% of the time), then A will shoot at C (30% hit rate)
  - If A misses, which is 70% of the time, then C will shoot B and B loses. (0.9)(0.7) = 63%
  - If A hits, then C is gone. Then B gets to shoot first in a duel with A. $\left(\frac{10}{37}\right)(0.3) \approx 8.1\%$.

Well, that's not great. But with a 10% hit rate, ending at 8.1% is decent. But just for kicks, let's check the other one.

Person B shoots at C first.

- If B hits C (10%), then B and A go back and forth in a duel. B wins about 19% of the time. Overall: (0.10)(0.19) = 0.019.
- If B misses C (90%), then C will shoot A immediately. Now B has one shot left to shoot at C (10%) win. Overall: (0.90)(0.10) = 0.09.

The best chance is if B shoots at C first.

WELL… ALMOST.

But Person B is sneaky and REALLY doesn’t want to do the toilet. Looking at the options, the best chance is to fire at C.

Notice how the best chance to win is when B shoots at C and misses. Ending result is about a 9% chance to win. That rate could go up if person B intentionally misses. At that point, C will shoot at A and hit, then B has a 10% chance to take out C.

Since 10% is the highest chance to win, the best chance for sneaky B is to point at C and then intentionally shoot the gun into the ground.

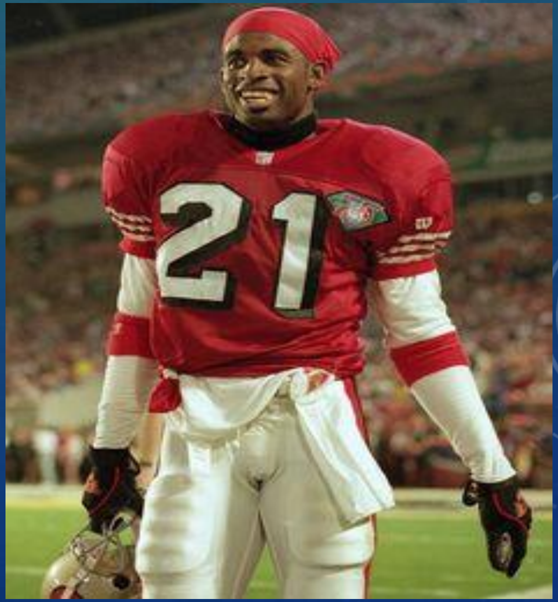
PRIME TIME …

To this point, we have devoted far too much time to Nerf and better hit some hard-core math soon. What is more hard-core than Prime Numbers!

Most of us know that primes are positive integers (natural numbers) that are divisible by only 1 and itself. This leads to the Fundamental Theorem of Arithmetic – that prime factorizations of numbers are unique. So 7 is prime because the only factors are 1 and 7, but 8 is not prime because it has a factor of 2. What do we call a number that is not prime?

- What are the first few primes?
- Is 1 prime? Why or why not?
- What is the 2nd prime number?

https://youtu.be/fEmSDHFYbms



Two numbers are called "relatively prime" if their only common factor is 1.

… AND A DRAWING

I now present my magic hat. In it I will place numbers and some of you will draw to see if we can find numbers that are relatively prime.

Key points:

- There are infinitely many prime numbers.
- There are infinitely many composite numbers.
- Relatively prime means the GCF or GCD is 1.

http://www.alcula.com/calculators/math/gcd/

Examples: start with 1, 2, 3, 4, 5. There are 10 different ways that we can choose 2 numbers: (12) (13) (14) (15) (23) (24) (25) (34) (35) (45) [You may think that there are 20 ways if the order matters, but order doesn't here and counting all 20 ways will get the same end result.]

How many of these pairs are relatively prime? The probability of drawing relatively prime is: ________

WE'LL NEED SOME VOLUNTEERS!

But if we add one more number and end with 1, 2, 3, 4, 5, 6, then there are 15 ways to select the numbers and 4 of them fail being relatively prime… still over 70%.

Let's try some. First, I'll put in the numbers 1 – 100. I'll need 3 volunteers.

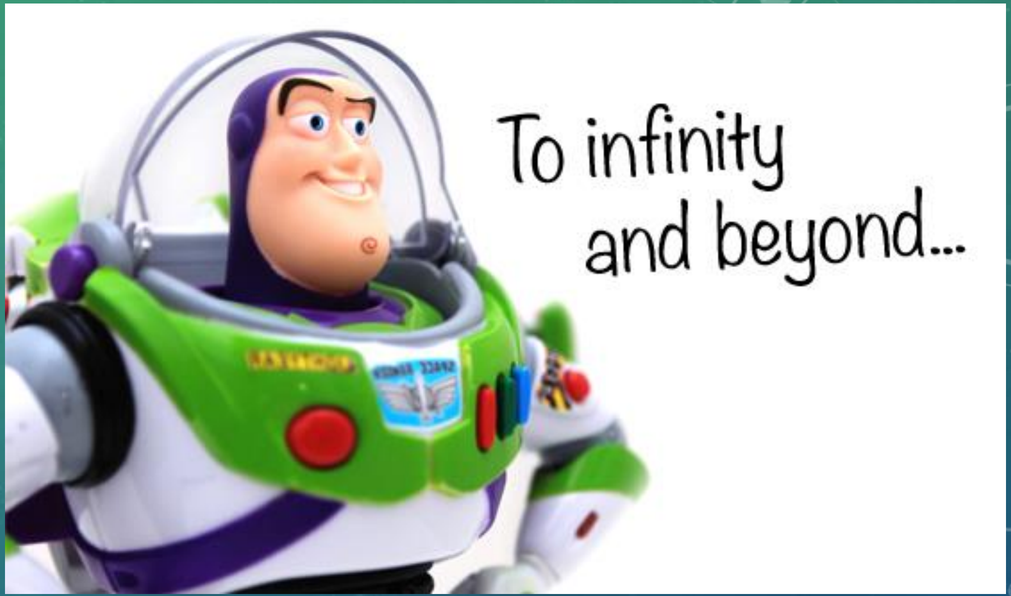
How did we do?

Next, I'll add the additional numbers from 101 – 200.

Check in #2.

Finally, I'll put in the additional numbers to have 1 – 1000 in the hat. Let's get 10 volunteers total and get a good idea of how this will turn out.

What if we put in all the natural numbers from 1 to infinity. All of them in one hat… then we draw. What is the probability that we draw two numbers and they are relatively prime?



Take your best guess now… as the numbers increase, there are fewer and fewer prime numbers.

Do you think that the ending probability will be:

A. Less than 40%
B. Between 40% and 60%
C. Between 60% and 70%
D. Between 70% and 80%
E. Over 80%
F. Not sure, but maybe it will involve this picture.

STOP – IN THE NAME OF MATH!

Alright, we need a bit of background from Math 30 – Beginning Algebra.

Back in that class, we saw $(1-x)(1+x) = 1 - x^2$.

What they didn't tell you is that this continues:

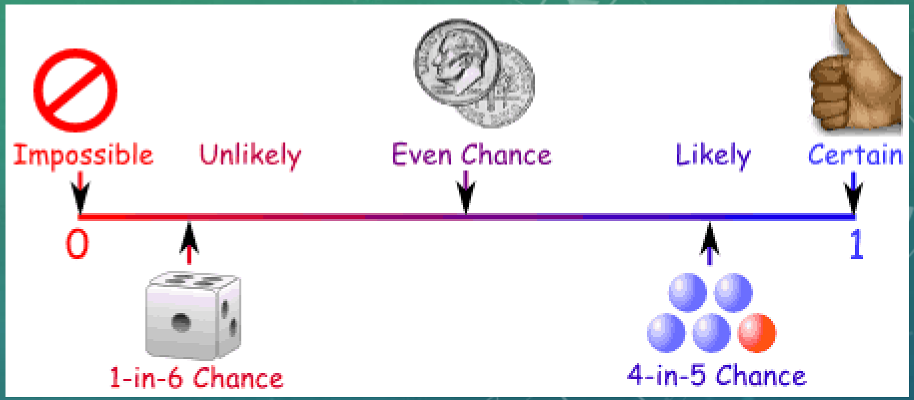
$$(1-x)(1+x+x^2) = 1 - x^3$$
$$(1-x)(1+x+x^2+x^3) = 1 - x^4$$
$$(1-x)(1+x+x^2+x^3+x^4) = 1 - x^5$$
$$(1-x)(1+x+x^2+x^3+x^4+\cdots) = 1, when\ |x| < 1$$

So this last one can also be written as:

$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$ or $\frac{1}{1+x+x^2+x^3+x^4+\cdots} = 1 - x$

This is a similar type of approach to the sum of a geometric sequence, which yields $S = \frac{a}{1-r}$.

GO ON… WHAT ARE THE CHANCES?



With that bit of math, we can continue.

We need to find the probability that two numbers share each Prime number.

- Probability that two numbers share a factor of 2 is ½ × ½. (half the numbers are even)
- So the probability that they don't share a factor of 2 is $\left(1-\frac{1}{2^2}\right)$.
- Likewise, the probability they don't share a factor of 3 is $\left(1-\frac{1}{3^2}\right)$.
- We don't need to worry about 4 because it isn't prime… so we'll skip to 5: $\left(1-\frac{1}{5^2}\right)$

We can continue this for all primes and use a property of probability to determine the probability two numbers do not share ANY prime:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots$$

Each one of these is like a $(1-x)$ from the previous page, so we could write each in expanded form since we know that $1-x=\frac{1}{1+x+x^2+x^3+x^4+\cdots}$.

LOOKS LIKE IT IS GETTING COMPLICATED!

Sometimes, we have to make things more complicated to simplify them… and the right hand side of $1 - x = \frac{1}{1+x+x^2+x^3+x^4+\cdots}$ does seem more complicated.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right) =$$

$$\frac{1}{\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^8}+\cdots\right)}\frac{1}{\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\frac{1}{3^8}+\cdots\right)}\frac{1}{\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\frac{1}{5^8}+\cdots\right)}\frac{1}{\left(1+\frac{1}{7^2}+\frac{1}{7^4}+\frac{1}{7^6}+\frac{1}{7^8}+\cdots\right)}\cdots$$

Phew. That looks better. ☺

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Phew. That looks better. ☺ Maybe not. You know what we should do: distribute it all out.

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Phew. That looks better. ☺ Maybe not. You know what we should do: distribute it all out. Oh Jeepers – it’s not that bad!

When you multiply it out, look at what you get:

$$\frac{1}{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}$$

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$$\frac{1}{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}$$

Told you it wasn't that bad.

SO NOW WHAT?

$\frac{1}{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}$ is our end probability, so if we can just find the value of the denominator, we'll be able to flip it over and be done. So what is the sum: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$?

We aren't the first to ask that question. It was first documented as being posed by Pietro Mengoli in 1644 and put in a book of his in 1650, and initially solved in 1735 with a rigorous proof in 1741. Who solved it?

It went to a group of specialists in Basel, Switzerland. They were the Bernoullis, Jacob and Johann, both prominent mathematicians.

- Jacob came up with the Bernoulli numbers and wrote one of the first books on probability. He wrote about this problem in 1689 starting the popularity!
- Johann worked on infinitesimal calculus… a lot.
    - His son, Daniel, created a principal in physics that later became the principal behind the Bernoulli Box – a high tech storage system from the mid-1980s that could contain 20MB or more of data in a removable diskette. Each was the size of a piece of paper. This box was made by Iomega, later making the "zip drive!"

THE BASEL PROBLEM

The Bernoulli brothers fought with this problem for decades, to no avail. Others who tried to solve it include Daniel Bernoulli, Gottfreid Leibniz (invented calculus), James Stirling (confirmed aspects of calculus), and Abraham de Moivre (trig/complex numbers, normal distribution and probability). Pretty much the biggest minds of the time. The struggle created many more areas of math and the concepts formed the basis for a lot of calculus… but there was no solution. And if a teacher can't seem to solve the problem, what do they do?

Assign it as homework for your students!! We can guarantee this happened, but one of the students of the Bernoulli brothers did solve it. His name was Euler. Leonhard Euler. (oiler)

Due to their location in Switzerland, this was known as the Basel Problem.
Leonhard Euler solved the problem when he was 28 (a perfect age!) but
the techniques he used weren't shown to be valid for another 6 years (perfect!).

[Perfect numbers are the sum of proper divisors: 6 = 1 + 2 + 3; 6 and 28 are perfect!]

So what did Euler do?

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So what did Euler do? Why he turned to trigonometry of course. This is the same Euler who created the Euler's formula that was used in my last talk to prove trigonometric identities. ☺

For those of you in Calculus 2, you know about using series to rewrite $\sin(x)$ using a Taylor Series Expansion: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

MATH MOMENT – HUMAN CALCULATORS

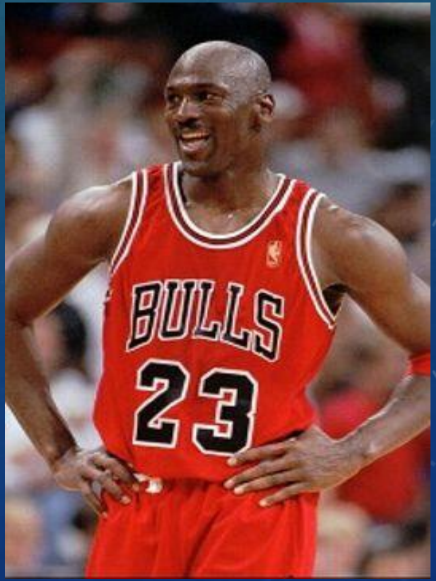
Tell me what you think when you see these numbers – show me your skills:

- 3.141592653589793238462643383…
- 1.57079632679489661923132169…
- 0.7853981633974483096156608…
- 0.3183098861837906715377675267…
- 6.28318530717958647692528676656…
- 0.86602540378443864676372317075…
- 1.41421356237309504880168872…
- 0.707106781186547524400844362…
- 1.7320508075688772935274463…
- 0.57735026918962576450914878…
- 2.718281828459045235360287471…
- 0.0769230769230769230769230769…
- 0.29411764705882352941176470588235…
- 0.2323232323232323232323232323…

MATH MOMENT – HUMAN CALCULATORS

- $3.14159265358979323846264338 3\ldots \approx \pi$
- $1.5707963267948966192313216 9\ldots \approx \frac{\pi}{2}$
- $0.7853981633974483096156608\ldots \approx \frac{\pi}{4}$
- $0.3183098861837906715377675267\ldots \approx \frac{1}{\pi}$
- $6.283185307179586476925286766 56\ldots \approx 2\pi$
- $0.8660254037844386467637231707 5\ldots \approx \frac{\sqrt{3}}{2}$
- $1.4142135623730950488016887 2\ldots \approx \sqrt{2}$
- $0.7071067811865475244008443 62\ldots \approx \frac{\sqrt{2}}{2}$
- $1.7320508075688772935274463\ldots \approx \sqrt{3}$
- $0.5773502691896257645091487 8\ldots \approx \frac{\sqrt{3}}{3}$
- $2.7182818284590452353602874 71\ldots \approx e$
- $0.0769230769230769230769230769\ldots = \frac{1}{13}$
- $0.29411764705882352941176470588235\ldots = \frac{5}{17}$
- $0.2323232323232323232323232323\ldots = \frac{23}{99}$ Michael Jordan as a repeating decimal fraction!

Michael Jordan needed to know how to shoot a free throw. He practiced so many times that it was easy. As you compute numbers over and over again, it becomes easier. Math teachers want students to have a number sense… because a number sense gives you an idea on how to solve a problem. But good number sense can help you see a solution… and how to get there.



HOW DID EULER GET TRIG FROM THIS PROBLEM?

http://eulerarchive.maa.org/hedi/HEDI-2003-12.pdf

James Sandifer did some research into this particular issue. Here's what he found out.

In order to solve this, Euler had to compute an integral that was extremely challenging, but Euler was very clever. He used an approximation technique that showed
$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \approx 1.644924.$

If you were to do the computation, you'd be out to 30,000 terms to get this level of accuracy.

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So where does the trigonometry come from? Well, here’s one cool thing about Euler – the kid could calculate. He recognized… RECOGNIZED… that this number was related to π.

Well not just π, it was a power of π… divided by another number. And that’s what gave him the idea.

Euler was the Michael Jordan of mathematics. He didn’t just play, he dominated!

BACK TO THE BASEL PROBLEM

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Those of you not in calculus may not yet have this knowledge, but just hang in there!

Next, Euler divided both sides by x : $\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$

Now he hit the pause button on calculus and turned to Pre-Calculus:

THE BASEL PROBLEM

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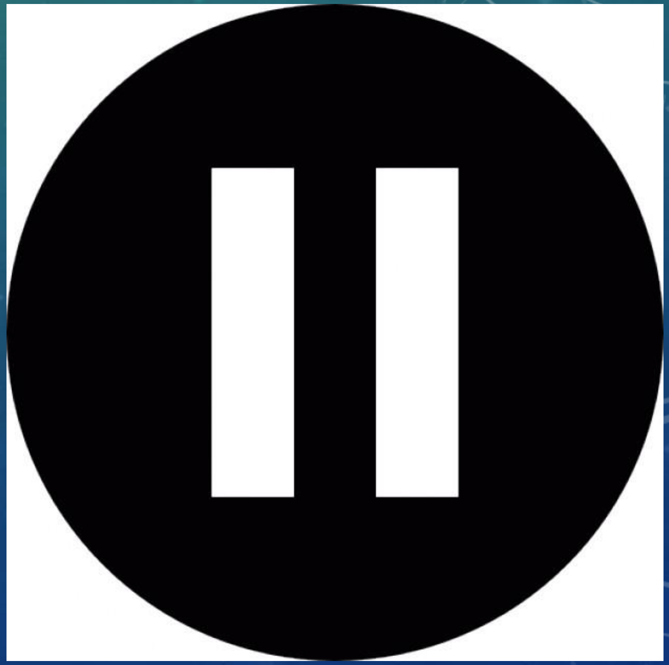

What is a function with roots of 2 and 5?

THE BASEL PROBLEM

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Now he hit the pause button on calculus and turned to Pre-Calculus:

What is a function with roots of 2 and 5?

$$f(x) = (x - 2)(x - 5)$$

So what about a function with roots of ± 2 and ± 5?

$$f(x) = (x - 2)(x + 2)(x - 5)(x + 5)$$

IT’S A SIN (CH).

The sine function has roots, but is not a polynomial. What are the roots of the sine function?

$0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \ldots$

The polynomial formed will be:

$$\sin(x) = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\ldots$$

So dividing by x and we get $\frac{\sin(x)}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\left(1-\frac{x^2}{4^2\pi^2}\right)\ldots$

Here’s the kicker, we can multiply these out and find all the x^2 terms:

$$-\frac{x^2}{\pi^2}-\frac{x^2}{2^2\pi^2}-\frac{x^2}{3^2\pi^2}-\frac{x^2}{4^2\pi^2}+\cdots$$

Next, Euler divided both sides by x : $\frac{\sin(x)}{x} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots$

IT'S A SIN (CH).

So from $\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{4^2\pi^2}\right)\ldots$, the coefficient on x^2 in " $-\frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} - \frac{x^2}{4^2\pi^2} + \cdots$ " is:

$$-\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \frac{1}{4^2\pi^2} + \cdots = -\frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

But Euler also had the other polynomial version: $\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$

The coefficient on x^2 in "$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$ " is: $-\frac{1}{3!} = -\frac{1}{6}$.

So $-\frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) = -\frac{1}{6}$, which means that $\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) = \frac{\pi^2}{6}$

WAIT, WHAT WERE WE TALKING ABOUT?

We know that $\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)=\frac{\pi^2}{6}$. But how does that help with the probability of picking two numbers that are relatively prime?

$\frac{1}{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}$ was our end probability, and if you substitute Euler's solution in here, you'll get the following:

$$\frac{1}{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}=\frac{1}{\left(\frac{\pi^2}{6}\right)}=\frac{6}{\pi^2}\approx 0.60792710185\ldots$$

Your chances of pulling two numbers that are relatively prime is a bit better than 60%.

WRAP UP – FOOD IMAGES

So this was pretty interesting. Some of you may have thought that the final solution involved this…

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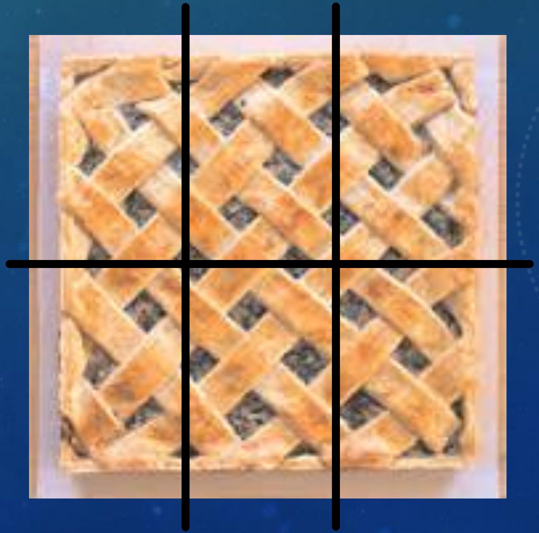
WRAP UP – MORE FOOD IMAGES

So this was pretty interesting. Some of you may have thought that the final solution involved this…

You've heard that pie are round, or the fun math joke extends this to say, "No way, pi r squared."

$A = \pi r^2$ from the area of a circle.

But in our probability question, it turned out that we were dealing with squared pi divided by 6. HA!



THANK YOU – I HOPE YOU ENJOYED THE TALK!

Future talks that could be put on:

- **The Mathematics of Elections: How a fair election is blown with one small Arrow**
- Graph Theory: Bridges and Salespeople… what to do?
- **Mathematics of Finance: Investments, loans, and retirement**
- The Mathematics of Card Shuffling: Patterns inside of Chaos!
- **Number Theory and Modular Arithmetic: Who needs negativity?**
- Seeing the Beauty of Pure Mathematics – Math for the sake of math.
- **Cutting a cake – How to mathematically distribute things fairly!**
- Game Theory – Showing you a beautiful mind